

LIGHT QUARK MASSES IN QCD

Johan Bijnens^a, Joaquín Prades^{a,b} and Eduardo de Rafael^c

^a NORDITA, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark

^b Niels Bohr Institute, Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark

^c Centre de Physique Théorique
CNRS-Luminy, Case 907
F-13288 Marseille Cedex 9, France

Abstract

We study the value of the light quark masses combination $m_u + m_d$ in QCD using both Finite Energy Sum Rules and Laplace Sum Rules. We have performed a detailed analysis of both the perturbative QCD and the hadronic parametrization inputs needed in these Sum Rules. As main result, we obtain $m_u(1 \text{ GeV}^2) + m_d(1 \text{ GeV}^2) = (12 \pm 2.5) \text{ MeV}$ for the running \overline{MS} masses.

1. The values of the light quark masses in the Standard Model, are still poorly known. We are referring to the masses of the u,d, and s quarks which appear in the QCD Lagrangian and which explicitly break the chiral-SU(3) flavour symmetry; the so called “current algebra” quark masses. These are the parameters which, among other things, generate the masses of the observed low-lying SU(3) octet of pseudoscalar particles i.e.; the Goldstone modes associated with the spontaneous breaking of the chiral-SU(3) symmetry. The precise relation between quark masses and pseudoscalar masses depends however on the magnitude of the various order parameters which govern the dynamics of the spontaneous chiral-SU(3) symmetry breaking. The order parameters are genuinely of non-perturbative origin, and this is what causes the difficulty in extracting the values of the light quark masses.

The masses of the π , K , and η particles are however smaller than the spontaneous chiral symmetry breaking scale Λ_χ , an empirical fact which justifies *a priori* a perturbative treatment of the problem in powers of quark masses. The appropriate field theoretical framework to carry out this expansion systematically is called Chiral Perturbation Theory (χPT) [1], [2], [3]. To lowest order in the expansion,

$$m_{\pi^+}^2 = m_{\pi^0}^2 = (m_u + m_d)\mathbf{B} + \mathcal{O}(m_q^2) \quad (1)$$

$$M_{K^+}^2 = (m_u + m_s)\mathbf{B} + \mathcal{O}(m_q^2) \quad (2)$$

$$M_{K^0}^2 = (m_d + m_s)\mathbf{B} + \mathcal{O}(m_q^2). \quad (3)$$

Ignoring the $\mathcal{O}(m_q^2)$ contributions, as well as electromagnetic corrections, the expansion to lowest order leads to the quark mass ratios estimates [4]:

$$\frac{m_d - m_u}{m_d + m_u} = \frac{M_{K^0}^2 - M_{K^+}^2 - (m_{\pi^0}^2 - m_{\pi^+}^2)_{\text{EM}}}{m_{\pi^0}^2} = 0.29; \quad (4)$$

and, with $\hat{m} \equiv (m_u + m_d)/2$,

$$\frac{m_s - \hat{m}}{2\hat{m}} = \frac{M_{K^0}^2 - m_{\pi^0}^2}{m_{\pi^0}^2} = 12.6. \quad (5)$$

The modification of these results due to the presence of higher order terms as well as to electromagnetic effects, has been an active field of research during the last few years. [For a recent review on the subject see ref. [5].] The χPT approach does not fix however the scale of the light quark masses. For example, the parameter \mathbf{B} which appears in eq. (1), is proportional to the size of the quark-antiquark vacuum condensate in the chiral limit. It is one of the fundamental coupling constants of the effective Lagrangian of QCD at low-energies; but its value cannot be determined by symmetry arguments alone. The appearance of extra symmetries in the effective Lagrangian [6], preventing the determination of the \mathbf{B} -like parameters, which appear in χPT , within χPT itself, reflects the same fact. Ideally, the numerical simulations of lattice QCD may fix the values of these \mathbf{B} -like parameters to a sufficiently reliable accuracy. [For a review of progress in this direction see e.g. ref. [7].] Here, we want to re-examine critically the possibility of fixing the value of the running masses $m_u(s) + m_d(s)$ at a specific energy scale s , using general analyticity properties of QCD two-point functions. We shall then come back to a discussion of several issues which have recently been raised in connection with the determination of the light quark masses.

2. The relevant two-point function for the determination of the combination of quark masses $m_u + m_d$ is the one associated with the divergence of the axial current with the quantum numbers of the pion field (see refs. [8] to [12],) i.e.;

$$\Psi_5(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T \{ \partial^\mu A_\mu^{(1-i2)}(x), \partial^\nu A_\nu^{(1+i2)}(0) \} | 0 \rangle, \quad (6)$$

where $[\bar{q} \equiv (\bar{u}, \bar{d}, \bar{s})$, and λ_i are Gell-Mann's flavour-SU(3) matrices,]

$$\partial^\mu A_\mu^{(1-i2)}(x) = (m_d + m_u) \bar{q}(x) i \gamma_5 \frac{\lambda_1 - i \lambda_2}{2} q(x) \quad (7)$$

In QCD, the function $\Psi_5(q^2)$ obeys a twice subtracted dispersion relation ($Q^2 = -q^2 \geq 0$ in our metric: $+- --$):

$$\Psi_5(q^2) = \Psi_5(0) + q^2 \Psi'_5(0) + Q^4 \int_0^\infty \frac{dt}{t^2} \frac{1}{t + Q^2} \frac{1}{\pi} \text{Im} \Psi_5(t). \quad (8)$$

The subtraction constants are governed by the low-energy behaviour of QCD. In particular

$$\begin{aligned} \Psi_5(0) &= -(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle \\ &\equiv 2f_\pi^2 m_\pi^2 (1 - \delta_\pi), \end{aligned} \quad (9)$$

where δ_π denotes higher order corrections. To $\mathcal{O}(p^4)$ in χPT [3] :

$$\delta_\pi = 4m_\pi^2 f_\pi^2 \frac{1}{\mathbf{f}_0^4} (2\mathbf{L}_8 - \mathbf{H}_2) + \mathcal{O}(p^6), \quad (10)$$

where \mathbf{f}_0 denotes the value of the pion coupling to vacuum in the chiral limit, in a normalization where the corresponding physical coupling is $f_\pi = 92.5$ MeV. The constants \mathbf{L}_8 and \mathbf{H}_2 are two specific coupling constants of the low-energy QCD chiral effective Lagrangian of $\mathcal{O}(p^4)$. The other constant $\Psi'_5(0)$ in eq.(8), brings in contributions of $\mathcal{O}(p^6)$ so far unknown, and therefore will not be used in our analysis.

At large Euclidean values of the momentum $Q^2 = -q^2$, the function Ψ_5 can be evaluated using perturbative QCD. The calculation of the first non-trivial order (i.e., two-loops,) was made in ref. [9], whereupon the first QCD analysis of the running light quark masses were made ([9], [10], [11], and [12].) The function Ψ_5 is now known at the three-loop level in perturbative QCD ([13] and [14]). Also, the present determinations of the QCD $\Lambda_{\overline{MS}}$ parameter (see ref. [15] for a recent review,) give values significantly larger than those used in the determination made in ref. [12]. These are two of the new ingredients which we shall incorporate in the present analysis.

The basic idea to obtain the value of $m_u + m_d$ is to use the dispersion relation in eq. (8) to relate the behaviour of the function Ψ_5 obtained from perturbative QCD, including some possible improvement from the knowledge of its leading non-perturbative $1/Q^2$ -power corrections [16] to specific integrals of the spectral function $\frac{1}{\pi} \text{Im} \Psi_5(t)$. The spectral function is an inclusive cross-section which, *ideally*, could be determined in terms of all the hadronic states with overall quantum numbers: $I^G(J^P) = 1^-(0^-)$, which the operator $\partial^\mu A_\mu^{(1-i2)}(x)$ can produce from the vacuum. The lowest hadronic state contributing to $\frac{1}{\pi} \text{Im} \Psi_5(t)$ is the pion pole. There is then a gap until one reaches the 3π -threshold. The shape of the low-energy end of the 3π -contribution can be determined from lowest order χPT [12]. One would

like however to incorporate the contribution from higher hadronic states as well. There are indeed states, like the $\pi(1300)$ [15], with the quantum numbers of the pion which have been observed in spectroscopic studies of the strong interactions. These states contribute also to the spectral function $\frac{1}{\pi}\text{Im}\Psi_5(t)$. However, as emphasized by the authors of ref. [17], their contribution brings in, *a priori*, other parameters than those measured in purely hadronic reactions. This is yet another issue which we shall also discuss in this new analysis.

3. A convenient way to encode the information provided by the dispersion relation in eq.(8) is to use a system of Finite Energy Sum Rules (FESR's). It has been shown that when the free hadronic parameters in a given spectral function are constrained by FESR's, the corresponding spectral function automatically satisfies the “heat evolution test” [18] of QCD-Hadronic duality. There are three FESR's which are relevant for our purposes:

$$\int_0^s \frac{dt}{t} \frac{1}{\pi} \text{Im}\Psi_5(t) = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 s \{1 + R_0(s)\} + \Psi_5(0); \quad (11)$$

$$\int_0^s dt \frac{1}{\pi} \text{Im}\Psi_5(t) = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 \frac{s^2}{2} \left\{ 1 + R_1(s) + 2 \frac{C_4 \langle O_4 \rangle}{s^2} \right\}; \quad (12)$$

$$\int_0^s dt t \frac{1}{\pi} \text{Im}\Psi_5(t) = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 \frac{s^3}{3} \left\{ 1 + R_2(s) - \frac{3}{2} \frac{C_6 \langle O_6 \rangle}{s^3} \right\}. \quad (13)$$

There is yet another sum rule with two inverse powers of t in the l.h.s. integrand which we could consider, but it brings in the unknown constant $\Psi'_5(0)$, and therefore it is not very useful. Other sum rules, with quadratic or higher powers of t , become more and more sensitive to the behaviour of the hadronic spectral function at high- t values, where there is little reliable experimental information at present.

The right hand sides of the three FESR's above are proportional to N_c , the number of QCD-colours. The functions $1 + R_n$ with $n = 0, 1, 2$ denote the corresponding QCD perturbative corrections,

$$\frac{-1}{2\pi i} \oint_s \frac{dt}{t} \left(1 - \frac{t}{s}\right) D_5(t) \quad (14)$$

with $D_5(t) \equiv t \frac{d}{dt} (\Psi_5(t)/t)$ for $n = 0$ and

$$\frac{-1}{2\pi i} \oint_s dt \left(1 - \frac{n+1}{n} \frac{t}{s} + \frac{1}{n} \left(\frac{t}{s}\right)^{n+1}\right) \Psi_5^{(2)}(t) \quad (15)$$

with $\Psi_5^{(2)}(t) \equiv \frac{d^2}{dt^2} \Psi_5(t)$ for $n = 1, 2$, once the sum of the running quark masses: $[m_u(s) + m_d(s)]^2$, evaluated at the two-loop level and $N_c/8\pi^2$ have been factored out. This $D_5(t)$ removes the unwanted subtraction constant $\Psi'_5(0)$ and $\Psi_5^{(2)}$ removes both $\Psi_5(0)$ and $\Psi'_5(0)$. Due to the truncated expressions we have for the series of the QCD perturbative corrections the explicit form of the R_n -functions depends on the specific choice of the renormalization scale [19]; as well as on whether or not the QCD-running of the two-point Ψ_5 -function over the circle of radius s in the complex t -plane is taken into account*. To check the numerical

*See ref. [20] for a discussion of the relevance of this dependence in the case of the hadronic width of the τ .

influence of the truncation in the QCD series for $\Psi_5(t)$ we shall consider two possible choices of the R_n -functions in the present analysis. One where we resum the leading and next-to-leading logs after the integral in eqs. (14) and (15) over the circle of radius s is done, by choosing the renormalization scale at $\nu^2 = s$, and another one where we first resum the leading and next-to-leading logs by choosing the renormalization scale at $\nu^2 = -t$, which we shall call “improved,” since it takes into account the possible large QCD-running of the two-point Ψ_5 -function over the circle of radius s . Other choices like $|\nu^2/s| \neq 1$ in the first case or $|\nu^2/t| \neq 1$ in the second choice do not improve the convergence of the perturbative QCD series for the Ψ_5 -function.

The quantities $C_4\langle O_4 \rangle$ and $C_6\langle O_6 \rangle$ in the FESR’s above, are a short-hand notation for the first non-perturbative power corrections of dimensions four and six. They are dominated respectively by the gluon condensate: $C_4\langle O_4 \rangle \simeq \frac{\pi}{N_c}\langle \alpha_s G^2 \rangle$, and the four-quark vacuum condensate which, in the vacuum saturation approximation [16], gives the estimate: $C_6\langle O_6 \rangle \simeq \frac{1792}{27N_c}\pi^3\alpha_s\langle \bar{q}q \rangle^2$, a result rather similar to the one obtained from the leading behaviour in the $1/N_c$ -expansion. In our numerical analysis we have allowed these non-perturbative parameters to have values within a rather generous range :

$$C_4\langle O_4 \rangle = (0.08 \pm 0.04)\text{GeV}^4, \quad (16)$$

$$C_6\langle O_6 \rangle = (0.04 \pm 0.03)\text{GeV}^6. \quad (17)$$

We use three active quark flavours in the QCD formulae and $\Lambda_{\overline{MS}}^{(3)} = 300 \pm 150 \text{ MeV}$ [15].

The procedure we use to extract $m_u + m_d$ from the FESR’s above, is the same as the one discussed in ref. [12]. First, one fixes the choice of the upper end value of s in the hadronic integrals, by demanding a good duality between the hadronic ratio of sum rules:

$$\mathcal{R}_{had.}(s) \equiv \frac{3}{2s} \frac{\int_0^s dt t \frac{1}{\pi} \text{Im} \Psi_5(t)}{\int_0^s dt t \frac{1}{\pi} \text{Im} \Psi_5(t)}, \quad (18)$$

and its QCD counterpart:

$$\mathcal{R}_{QCD}(s) \equiv \frac{1 + R_3(s) - \frac{3}{2} \frac{C_6\langle O_6 \rangle}{s^3}}{1 + R_2(s) + 2 \frac{C_4\langle O_4 \rangle}{s^2}}. \quad (19)$$

With s fixed within the duality region, one then solves for $m_u + m_d$ using the second sum rule in (12); and for δ_π , defined in eq.(10), from the first sum rule in (11).

4. Another technique to exploit the information encoded in the dispersion relation in eq.(8), consists in using sum rules which relate moments of the Laplace transform of the hadronic spectral function to their QCD-counterparts. Here, the master equation is [10]

$$\begin{aligned} \frac{1}{(M^2)^2} \int_0^\infty dt e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im} \Psi_5(t) &= \frac{N_c}{8\pi^2} [m_u(M^2) + m_d(M^2)]^2 \\ &\times \left\{ 1 + \Delta(M^2) + 2 \frac{C_4\langle O_4 \rangle}{M^4} + \frac{3}{2} \frac{C_6\langle O_6 \rangle}{M^6} \right\}, \end{aligned} \quad (20)$$

where $\Delta(M^2)$ denotes the appropriate QCD perturbative corrections, once the overall running mass dependence at the M^2 -scale has been factored out; and the leading and next to leading non-perturbative $1/M^2$ -power corrections are explicitly shown. Here again, one has to find first a window of duality in the M^2 -variable. Usually, this is fixed from the comparison of the M^2 -dependence of the ratio of hadronic moments:

$$\mathcal{L}(M^2, s_0) \equiv \frac{\int_0^{s_0} dt t e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im}\Psi_5(t)}{\int_0^{s_0} dt e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im}\Psi_5(t)}, \quad (21)$$

to the M^2 -dependence of its QCD-counterpart. Then, one solves for the quark masses using eq.(20), with M^2 fixed in the duality region. Because of the exponential factor, the integral in the l.h.s. of eq.(20) weighs significantly the low energy region of the hadronic spectral function. On the other hand, the Laplace transform itself extends from 0 to ∞ . In practice, this means that from a certain s_0 -threshold onwards, the hadronic spectral function will be identified with its QCD perturbative counterpart. This threshold choice brings in a new parameter to be fixed. As shown in ref. [18], the appropriate choice of the onset of the perturbative continuum, corresponds in fact to the upper end of the duality region which one gets using the FESR's; i.e., $s_0 \simeq s$. In the present analysis, we shall use the Laplace transform technique only as an overall check of the determination of the value of $m_u + m_d$ obtained from the FESR's. Here we have only used the scaling at $\nu^2 = s$. The renormalization scale choice dependence due to the truncation of the QCD perturbative series is similar to the one found in the FESR's analysis.

5. We shall now discuss the hadronic phenomenological input for the spectral function $\frac{1}{\pi} \text{Im}\Psi_5(t)$ which appears in the integrand in the l.h.s. of the FESR's in eqs.(11), (12), and (13).

As already said before, the lowest hadronic state contributing to $\frac{1}{\pi} \text{Im}\Psi_5(t)$ is the pion pole. There is then a gap until one reaches the 3π -threshold. The shape of the low-energy end of the 3π -threshold can be calculated using lowest order χPT , with the result, $[\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca]$:

$$\frac{1}{\pi} \text{Im}\Psi_5(t) = 2f_\pi^2 m_\pi^4 \delta(t - m_\pi^2) + \theta(t - 9m_\pi^2) \frac{2f_\pi^2 m_\pi^4}{(16\pi^2 f_\pi^2)^2} \frac{t}{18} \rho_\chi^{3\pi}(t); \quad (22)$$

with [†]

$$\begin{aligned} \rho_\chi^{3\pi}(t) &= \int_{4m_\pi^2}^{(\sqrt{t}-m_\pi)^2} \frac{du}{t} \sqrt{1 - \frac{4m_\pi^2}{u}} \left\{ 5 + \frac{1}{2} \frac{1}{(t - m_\pi^2)^2} \left[(t - 3(u - m_\pi^2))^2 \right. \right. \\ &+ \left. \left. 3\lambda(t, u, m_\pi^2) \left(1 - \frac{4m_\pi^2}{u}\right) + 20m_\pi^4 \right] + \frac{1}{(t - m_\pi^2)} \left[3(u - m_\pi^2) - t + 9m_\pi^2 \right] \right\}. \end{aligned} \quad (23)$$

Notice that the chiral power counting of the pion pole term contribution to $\frac{1}{\pi} \text{Im}\Psi_5(t)$ is $\mathcal{O}(p^2)$; while the contribution from the χPT 3π -continuum is $\mathcal{O}(p^6)$. From the point of view of the large- N_c counting rules, the pion pole contribution is leading; i.e., $\mathcal{O}(N_c)$, while the

[†]The corresponding expression in eq.(4.14) of ref. [12] has several misprints which we have corrected here.

χPT 3π -continuum is suppressed down to $\mathcal{O}(1/N_c)$. Because of the existence of resonance states, we expect, however, the three pion continuum to be enhanced, and contribute to $\mathcal{O}(N_c)$ as well. The question is how to fold the observed π' resonance states to the spectral function above. The authors of ref. [17] have shown how, ideally, one could measure directly the spectral function we are interested in from very high statistics analyses of $\tau \rightarrow \nu_\tau + 3\pi$ decays. In the absence of this direct experimental information, it is important to discuss how these $\pi'(0^{++})$ -states can contribute, and how to propose an hadronic ansatz for $\frac{1}{\pi}\text{Im}\Psi_5(t)$ as model independent as possible.

The situation is in fact rather similar to the one already encountered in the more familiar case of the vector-vector correlation function with the quantum numbers of the ρ . In this case, the relevant two-point function obeys a once subtracted dispersion relation:

$$\Pi(q^2) = \Pi(0) + q^2 \int_0^\infty \frac{dt}{t} \frac{1}{t - q^2} \frac{1}{\pi} \text{Im}\Pi(t), \quad (24)$$

and to lowest order in χPT ($\mathcal{O}(p^4)$ in this case,) the spectral function obtained from the 2π -cut discontinuity is:

$$\frac{1}{\pi} \text{Im}\Pi(t) = \frac{1}{16\pi^2} \frac{1}{3} \left(1 - \frac{4m_\pi^2}{t}\right)^{\frac{3}{2}} \theta(t - 4m_\pi^2). \quad (25)$$

Again, this contribution is non-leading from the point of view of the large- N_c counting rules, but we know that this is not the full story. The ρ -resonance also contributes to $\frac{1}{\pi}\text{Im}\Pi(t)$, and its contribution is indeed leading [$\mathcal{O}(N_c)$] in the large- N_c limit. Knowing the mass M_V and the width Γ_V (i.e., the $\rho \rightarrow 2\pi$ coupling g_V) from strong interaction experiments, is not enough, *a priori*, to fix the contribution of the same ρ -resonance to the spectral function $\frac{1}{\pi}\text{Im}\Pi(t)$. Indeed, at the narrow width approximation, the ρ -contribution to the spectral function:

$$\frac{1}{\pi} \text{Im}\Pi(t) = f_V^2 M_V^2 \pi \delta(t - M_V^2), \quad (26)$$

brings in the coupling f_V of the ρ to the external vector-isovector current, which is different to the coupling g_V . These two coupling constants can however be related under the rather reasonable assumption of an unsubtracted dispersion relation for the electromagnetic form factor of the pion, and ρ -dominance of the form factor discontinuity. Then, the evaluation of the real part of the form factor at $q^2 = 0$, constrains the two couplings to satisfy the relation [21]:

$$1 = f_V g_V \frac{M_V^2}{f_\pi^2}. \quad (27)$$

The same relation follows, if we parametrize the spectral function $\frac{1}{\pi}\text{Im}\Pi(t)$ with a Breit-Wigner shape function, and we fix the overall normalization to coincide, at the 2π -threshold, with the one provided by the lowest order χPT -prediction:

$$\frac{1}{\pi} \text{Im}\Pi(t) = \theta(t - 4m_\pi^2) \frac{1}{16\pi^2} \frac{1}{3} \left(1 - \frac{4m_\pi^2}{t}\right)^{\frac{3}{2}} \frac{(M_V^2 - 4m_\pi^2)^2 + M_V^2 \Gamma_V^2}{(M_V^2 - t)^2 + M_V^2 \Gamma_V^2}. \quad (28)$$

Again, at the narrow width approximation

$$\frac{1}{(M_V^2 - t)^2 + M_V^2 \Gamma_V^2} \Rightarrow \frac{1}{M_V \Gamma_V} \pi \delta(t - M_V^2), \quad (29)$$

and assuming that in the limit of massless pions Γ_V is dominated by the 2π -mode, results in the expression

$$\frac{1}{\pi} \text{Im}\Pi(t) = \frac{f_\pi^4}{g_V^2 M_V^2}. \quad (30)$$

The identification of the two narrow width expressions (26) and (30) leads then to the same relation as in eq.(27).

It is well known that the hadronic parametrization in eq.(28) is in fact in very close agreement with the low-energy phenomenological spectral function extracted from the e^+e^- annihilation cross-section data. Following this analogy, we propose to parametrize the 3π -continuum contribution to $\frac{1}{\pi} \text{Im}\Psi_5(t)$, modifying eq.(23) as follows:

$$\rho_\chi^{3\pi}(t) \Rightarrow \rho_{had.}(t), \quad (31)$$

with

$$\begin{aligned} \rho_{had.}(t) &= |F(M_1, \Gamma_1; M_2, \Gamma_2; \xi; t)|^2 \int_{4m_\pi^2}^{(\sqrt{t}-m_\pi)^2} \frac{du}{t} \sqrt{1 - \frac{4m_\pi^2}{u}} \\ &\times \left\{ 5 + \frac{1}{2} \frac{1}{(t - m_\pi^2)^2} \left[(t - 3(u - m_\pi^2))^2 + 20m_\pi^4 \right. \right. \\ &+ 3\lambda(t, u, m_\pi^2) \left(1 - \frac{4m_\pi^2}{u}\right) \frac{(M_\rho^2 - 4m_\pi^2)^2 + M_\rho^2 \Gamma_\rho^2}{(M_\rho^2 - u)^2 + M_\rho^2 \Gamma_\rho^2} \Big] \\ &+ \left. \frac{1}{(t - m_\pi^2)} \left[3(u - m_\pi^2) - t + 9m_\pi^2 \right] \right\}. \end{aligned} \quad (32)$$

The function F which appears as an overall factor in the r.h.s. encodes the modulation due to the presence of two π' -resonances[‡] with possible interference (ξ is a complex parameter,) and with masses $M_{1,2}$ and widths $\Gamma_{1,2}$:

$$F[M_1, \Gamma_1; M_2, \Gamma_2; \xi; t] = \frac{\left| \frac{1}{(t - M_1^2) + i\Gamma_1 M_1} + \xi \frac{1}{(t - M_2^2) + i\Gamma_2 M_2} \right|^2}{\left| \frac{1}{(9m_\pi^2 - M_1^2) + i\Gamma_1 M_1} + \xi \frac{1}{(9m_\pi^2 - M_2^2) + i\Gamma_2 M_2} \right|^2}. \quad (33)$$

In the hadronic parametrization above we have also allowed for a possible modulation of the (1^-) 2π -subchannel in the 3π -continuum to take into account the effect of the $\rho(770)$ -resonance. The factor F has been normalized to 1 at the 3π -threshold, but we shall also discuss the effect of possibly larger normalization values as suggested in ref. [17].

6. The shape of the hadronic continuum contribution to the spectral function in (22) i.e., the function

[‡]A possible, not yet confirmed, new π' -resonance has been reported in [22]. However its inclusion in the present analysis does not affect any of the results.

$$\frac{1}{\pi} \text{Im} \Psi_5(t) \Big|_{\text{HC}} \equiv \theta(t - 9m_\pi^2) \frac{2f_\pi^2 m_\pi^4}{(16\pi^2 f_\pi^2)^2} \frac{t}{18} \rho_{\text{had.}}(t); \quad (34)$$

has been plotted in Fig.1 for s -values between the 3π -threshold and 4 GeV^2 . The masses and widths of the π' -resonances have been fixed to the estimated values in ref. [15] (i.e., in MeV units:)

$$M_1 = 1300 \pm 100, \Gamma_1 = 400 \pm 200; \quad M_2 = 1770 \pm 30, \Gamma_2 = 310 \pm 50. \quad (35)$$

The three curves in Fig.1 correspond to the ξ -parameter values: $\xi = 0.234 + i0.1$ (the dotted curve); $\xi = -0.23 + i0.65$ (the continuous curve) and $\xi = 0.4 + i0.4$ (the dashed curve) where the normalization at the 3π -threshold is twice as large as the one in eq. (33). The first ξ -value (dotted curve) is the one which reproduces the best fit to the experimental curves which observe the $\pi'(1770)$ in hadronic interactions [23]. The second ξ -value (continuous curve) is the one which gives the best duality with QCD. The third ξ -value is one which gives a good duality with QCD with a normalization at the 3π -threshold equal to 2. Other normalizations lead to similar results for the quark masses after imposing the QCD-Hadronic duality constraint.

The duality test is shown in Fig.2. Here, the short dashed curve is the one corresponding to the QCD ratio $\mathcal{R}_{QCD}(s)$ in eq.(19) and the dash-dotted one is the same ratio evaluated with the “improved” set of $R_{1,2}$ -functions discussed above. Most of the difference between the perturbative and the “improved” version is in R_2 and very little in R_1 . This explains why the quark masses obtained show less variation than the duality ratio. The other curves are the hadronic ratios, with the same input as in Fig. 1. As can be seen curve 1 has no good duality but 2 and 3 are acceptable. We show it further as an illustration of the type of variation obtained with the hadronic input.

Figure 3 shows the results we get for the quark masses. Here we show the value we get for the sum of running masses $m_u(1\text{GeV}^2) + m_d(1\text{GeV}^2)$ in MeV in the \overline{MS} -scheme. The curves correspond to the result we obtain solving for $m_u + m_d$ in the FESR in eq.(12), using the “improved” expression for the QCD-function $R_1(s)$, and using the three hadronic parametrizations shown in Fig.1. The continuous curve corresponds to the hadronic parametrization which passed best the QCD-Hadronic duality test. For this case we also show the result with the perturbative expression for R_1 . The stability of the result is rather remarkable. If the same good QCD-Hadronic duality is wanted, then a modification of the hadronic parametrization of the 3π -continuum in (32), including, for instance, a global normalization factor at the 3π -threshold to the two π' -resonances overall function F in (33) (as suggested in ref. [17]), or varying the complex parameter ξ , leads to a spectral function very similar to the continuous curve in Fig. 1 and therefore to the same results from the FESR's. As an example we have shown the effect of a normalization factor of 2 and a variation of ξ . Thus, we quote the results for the best duality hadronic parametrization:

$$m_u(1\text{GeV}^2) + m_d(1\text{GeV}^2) = (12 \pm 2.5) \text{ MeV}, \quad (36)$$

where the error reflects changes in the input parameters.

7. It is well known that the scalar channels can be affected by instanton effects, however, there is no clear computational scheme to get a reliable estimate of these effects (for an

attempt see [24]). Nevertheless, it is also obvious from the results in this reference that if the duality region is for energies around $s \simeq (2.5 \sim 3.5) \text{ GeV}^2$ as we have, these effects can be absorbed in the quoted error bars for the masses above from the FESR analysis.

Figure 4 shows the result we get for the parameter δ_π defined in eq.(9). This is the result of solving for δ_π in the first FESR in (11). Again the continuous curve is the one corresponding to the hadronic parametrization which passed best the QCD-Hadronic duality test i.e., the one which leads to the continuous lines also in Figs.1 and 3. For the QCD function R_0 we use the perturbative expression. The improved one is within the errors. The stability of the result is less impressive than for the quark masses in Fig. 3, but good enough (notice the vertical scale,) to obtain an estimate of this parameter.

$$\delta_\pi = (3.5 \pm 1)\%. \quad (37)$$

From this result and using (10) we can obtain for the scale independent quantity [3] $2\mathbf{L}_8 - \mathbf{H}_2$

$$2\mathbf{L}_8 - \mathbf{H}_2 = (2.9 \pm 1.0) \times 10^{-3}, \quad (38)$$

and from (9)

$$\frac{\langle \bar{u}u + \bar{d}d \rangle (1\text{GeV}^2)}{2} = -(0.013 \pm 0.003)\text{GeV}^3. \quad (39)$$

This result is for the non-normal ordered quark condensate (see refs. [25] and [26].)

We can also combine our result for the light quark masses in (36) with the recent results for the strange quark mass in refs. [25,26] and [7] to get the ratio of quark masses (we use $m_s(1 \text{ GeV}^2) = (175 \pm 25) \text{ MeV}$):

$$r \equiv \frac{m_s}{\hat{m}} = 29 \pm 7. \quad (40)$$

Although the error bars are large, this result is free of the uncertainties noted in [6] since it is obtained from QCD.

We can also combine our result with the estimation of the next-to-leading corrections to Dashen's theorem in refs. [27] and [28]

$$\left(M_{K^+}^2 - M_{K^0}^2 \right)_{\text{EM}} = (1.9 \pm 0.4) \left(m_{\pi^+}^2 - m_{\pi^0}^2 \right)_{\text{EM}}. \quad (41)$$

This result translates into, using the ratio from [3],

$$\frac{m_d - m_u}{m_d + m_u} = \frac{m_\pi^2}{M_K^2} \frac{(M_{K^0}^2 - M_{K^+}^2)_{\text{QCD}}}{M_K^2 - m_\pi^2} \frac{m_s^2 - \hat{m}^2}{4\hat{m}^2} = (0.52 \pm 0.05) \times 10^{-3} (r^2 - 1) \quad (42)$$

and with the values for the quark mass ratio r above we get

$$\frac{m_u}{m_d} = 0.44 \pm 0.22. \quad (43)$$

We shall next discuss the consistency of our results for the quark masses, using FESR's, with the results from the Laplace transform technique. With the replacement in eq. (31) incorporated in the spectral function in (22), the master equation in (20) can be written as

$$\begin{aligned} \frac{2f_\pi^2 m_\pi^4}{M^4} \left\{ e^{-\frac{m_\pi^2}{M^2}} + \frac{1}{(16\pi^2 f_\pi^2)^2} \frac{1}{18} \int_{9m_\pi^2}^s dt e^{-\frac{t}{M^2}} \rho_{had}(t) \right\} &= \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 \\ &\times \left\{ 1 + P(M^2, s) + 2 \frac{C_4 \langle O_4 \rangle}{M^4} + \frac{3}{2} \frac{C_6 \langle O_6 \rangle}{M^6} \right\}, \end{aligned} \quad (44)$$

with $P(M^2, s)$ the corresponding perturbative QCD expression given in the appendix. Solving for the quark masses in this expression results in the plots shown in Fig. 5. As in the previous figures we show the three hadronic parametrizations of Fig. 1 for $s = 2.5 \text{ GeV}^2$. The results for $s = 3.5 \text{ GeV}^2$ are very similar. The agreement between the two determinations is very good. We have also checked that the hadronic parametrization gives a reasonable duality for the ratio in eq. (21).

Acknowledgements

EdeR thanks NORDITA where part of this work was done for hospitality. JP thanks the Leon Rosenfeld foundation (Københavns Universitet) for support and CICYT (Spain) for partial support under Grant No AEN-93-0234. This work was also done under NorFA grant No. 93.15.078/00 and INTAS 93-283.

Appendix

1. THE TWO-POINT FUNCTION $\Psi_5(q^2)$ IN PERTURBATION THEORY.

With ν^2 the renormalization-scale, and $Q^2 = -q^2$:

$$\begin{aligned} \Psi_5(q^2) \Big|_{QCD} &= \frac{N_c}{8\pi^2} [m_u(\nu^2) + m_d(\nu^2)]^2 Q^2 \left\{ b_0 \log \frac{Q^2}{\nu^2} + a_0 \right. \\ &+ \frac{\alpha_s(\nu^2)}{\pi} \left[c_1 \log^2 \frac{Q^2}{\nu^2} + b_1 \log \frac{Q^2}{\nu^2} + a_1 \right] \\ &+ \left(\frac{\alpha_s(\nu^2)}{\pi} \right)^2 \left[d_2 \log^3 \frac{Q^2}{\nu^2} + c_2 \log^2 \frac{Q^2}{\nu^2} + b_2 \log \frac{Q^2}{\nu^2} + a_2 \right] \\ &\left. + \mathcal{O} \left[\left(\frac{\alpha_s}{\pi} \right)^3 \right] \right\}. \end{aligned} \quad (A.1)$$

The non-trivial coefficients are $b_0 = 1$; $b_1 = 17/3$ and

$$b_2 = \frac{10801}{144} - \frac{39}{2} \zeta(3) - \left(\frac{65}{24} - \frac{2}{3} \zeta(3) \right) n_f, \quad (A.2)$$

with n_f the number of active quark flavours. The other coefficients are fixed by renormalization group properties with the results: $c_1 = -\gamma_1/2$; $c_2 = (b_1/4)(\beta_1 - 2\gamma_1) - \gamma_2/2$ and $d_2 = (-\gamma_1/12)(\beta_1 - 2\gamma_1)$, where β_i and γ_i are the coefficients of the perturbation theory series

expansion in powers of $\alpha_s(\nu^2)/\pi$ of the $\beta(\alpha_s)$ -function associated with the QCD coupling constant renormalization: $\beta_1 = -11/2 + n_f/3$ and $\beta_2 = -51/4 + 19n_f/12$; and the $\gamma(\alpha_s)$ -function associate with mass renormalization: $\gamma_1 = 2$ and $\gamma_2 = 101/12 - 5n_f/18$. From eq. (A.1) we can easily obtain the expressions for $D_5(t)$ and $\Psi_5^{(2)}(t)$ with $t = q^2$, needed in eqs. (14) and (15) for the evaluation of the FESR's.

2. THE SPECTRAL FUNCTION $\frac{1}{\pi}\text{Im}\Psi_5(t)$ IN PERTURBATION THEORY.

$$\begin{aligned} \frac{1}{\pi}\text{Im}\Psi_5(t)\Big|_{\text{QCD}} &= \frac{N_c}{8\pi^2}[m_u(\nu^2) + m_d(\nu^2)]^2 t \left\{ 1 + \frac{\alpha_s(\nu^2)}{\pi} \left[b_1 + 2c_1 \log \frac{t}{\nu^2} \right] \right. \\ &+ \left(\frac{\alpha_s(\nu^2)}{\pi} \right)^2 \left[b_2 + 2c_2 \log \frac{t}{\nu^2} + d_2(3 \log^2 \frac{t}{\nu^2} - \pi^2) \right] \\ &+ \left. \mathcal{O} \left[\left(\frac{\alpha_s}{\pi} \right)^3 \right] \right\}. \end{aligned} \quad (\text{A.3})$$

3. THE RUNNING COUPLING CONSTANT AND THE RUNNING MASS.

With $a_1 \equiv 2/(-\beta_1 \log \frac{Q^2}{\Lambda^2})$, the running coupling at the two-loop level, at the scale Q^2 , is

$$a(Q^2) \equiv \frac{\alpha_s(Q^2)}{\pi} = a_1 \left\{ 1 - a_1 \frac{\beta_2}{\beta_1} \log \log \frac{Q^2}{\Lambda^2} \right\}. \quad (\text{A.4})$$

At that level, the renormalization group summed expression for the running mass at the Q^2 -scale is:

$$m(Q^2) = \overline{m} \left[a(Q^2) \right]^{\frac{\gamma_1}{-\beta_1}} \left[1 + \frac{\beta_2}{\beta_1} a(Q^2) \right]^{\left(\frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right)}, \quad (\text{A.5})$$

where \overline{m} fixes the arbitrary constant of motion of the renormalization flow differential equation, which defines an invariant mass.

4. THE LAPLACE TRANSFORM $\mathcal{M}_5(M^2)$ IN PERTURBATION THEORY.

The operator \hat{L} which transforms $\Psi_5^{(2)}(Q^2)$ into the function

$$\mathcal{M}_5(M^2) = \frac{1}{(M^2)^3} \int_0^\infty dt e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im}\Psi_5(t), \quad (\text{A.6})$$

is

$$\hat{L} \equiv \lim_{N \rightarrow \infty} \left\{ \frac{(-Q^2)^N}{\Gamma(N)} \frac{\partial^N}{(\partial Q^2)^N} \right\} \quad (\text{A.7})$$

with $M^2 \equiv Q^2/N$ finite.

As explained in the text, in practice we have identified the hadronic spectral function $\frac{1}{\pi}\text{Im}\Psi_5(t)$ in the r.h.s. of eq. (A.6) with its perturbative QCD expression from some threshold $s_0 \simeq s$ onwards. This allows us to rewrite eq. (20) as

$$\begin{aligned} \frac{1}{(M^2)^2} \int_0^s dt e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im}\Psi_5(t) &= \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 \\ &\times \left\{ 1 + P(M^2, s) + 2 \frac{C_4 \langle O_4 \rangle}{M^4} + \frac{3}{2} \frac{C_6 \langle O_6 \rangle}{M^6} \right\} \quad (\text{A.8}) \end{aligned}$$

with $P(M^2, s)$ defined by

$$\frac{1}{(M^2)^2} \int_0^s dt e^{-\frac{t}{M^2}} \frac{1}{\pi} \text{Im}\Psi_5(t) \Big|_{QCD} = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 \left\{ 1 + P(M^2, s) \right\}. \quad (\text{A.9})$$

References

- [1] S. Weinberg, *Physica* A96 (1979) 327.
- [2] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* 158 (1984) 142.
- [3] J. Gasser and H. Leutwyler, *Nucl. Phys. B*250 (1985) 465, 517, 539.
- [4] S. Weinberg, *Festschrift for I.I. Rabi*, New York Acad. Sci. (N.Y.) (1978).
- [5] H. Leutwyler, *Masses of the Light Quarks*, Bern preprint BUTP-94/8
- [6] D.B. Kaplan and A.V. Manohar, *Phys. Rev. Lett.* 56 (1986) 2004;
H. Georgi and I. McArthur, Harvard Univ. Report No HUTP-81/A011.
- [7] Proc. of the LATTICE93 Conference, *Nucl. Phys. B (Proc. Suppl.)* 34 (1994) ;
C.R. Allton, M. Ciuchini, M. Crisafulli, E. Franco, V. Lubicz and G. Martinelli, *Quark Masses from Lattice QCD at the Next-to-Leading Order*, ROME preprint 94/1018, CERN-TH.7266/94.
- [8] A.I. Vainshtein, M.B. Voloshin, V.I. Zakharov, V.A. Novikov, L.B. Okun', and M.A. Shifman, *Sov. J. Nucl. Phys.* 27(2), (1978) 274.
- [9] C. Becchi, S. Narison, E. de Rafael, and F.J. Ynduráin, *Z. Phys.* C8 (1981) 335.
- [10] S. Narison and E. de Rafael, *Phys. Lett.* B103 (1981) 57.
- [11] E. de Rafael, *Current Algebra Quark Masses in QCD*, Proceedings of the NSF-CNRS joint seminar on Recent Developments in QCD, June 1981 (J. Dash, Ed.) Marseille preprint CPT-81/1344.
- [12] C.A. Dominguez and E. de Rafael, *Ann. Phys. (N.Y.)* 174 (1987) 372.
- [13] S.G. Gorishny, A.L. Kataev, and S.A. Larin, *Phys. Lett.* B135 (1984) 457.
- [14] S.G. Gorishny, A.L. Kataev, S.A. Larin, and L.R. Surguladze, *Mod. Phys. Lett.* A5 (1990) 2703.
- [15] Review of Particle Properties, *Phys. Rev. D*50 (1994) 1173.

- [16] M.A. Shifman, A.I. Vainshtein, and V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.
- [17] J. Stern, N.H. Fuchs, and M. Knecht, *Light Quark Masses from Exclusive Tau decays: An Experimental Proposal*. To appear in the Proc. of the Third Workshop on the τ Charm Factory. Orsay preprint IPNO/TH 93-38.
- [18] R.A. Bertlmann, G. Launer, and E. de Rafael, Nucl. Phys. B250 (1985) 61.
- [19] P.M. Stevenson, Phys. Rev. D23 (1981) 2916.
- [20] F. Le Diberder and A. Pich, Phys. Lett. B286 (1992) 147.
- [21] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, and E. de Rafael, Phys. Lett. B223 (1989) 425.
- [22] Yu.I. Ivanshin, V.A. Petrov, A.A. Tyapkin, and I.M. Vasilevsky, *Evidence for a new $0^- S$ resonance in the diffractively produced 3π -system*. JINR preprint E1-93-155 (1993).
- [23] G. Bellini et al., Nuovo Cim. 79A (1984) 282.
- [24] E. Gabrielli and P. Nason, Phys. Lett. B313 (1993) 430.
- [25] M. Jamin and M. Münz, *The strange quark mass from QCD sum rules*. CERN-TH.7435/94.
- [26] K.G. Chetyrkin, C.A. Dominguez, D. Pirjol, and K. Schilcher *Mass singularities in light quark correlators: the strange quark case*. Mainz preprint MZ-TH/94-21.
- [27] J. Bijnens, Phys. Lett. B306 (1993) 343.
- [28] J.F. Donoghue, B.R. Holstein, and D. Wyler, Phys. Rev. D47 (1993) 2089.

Figures

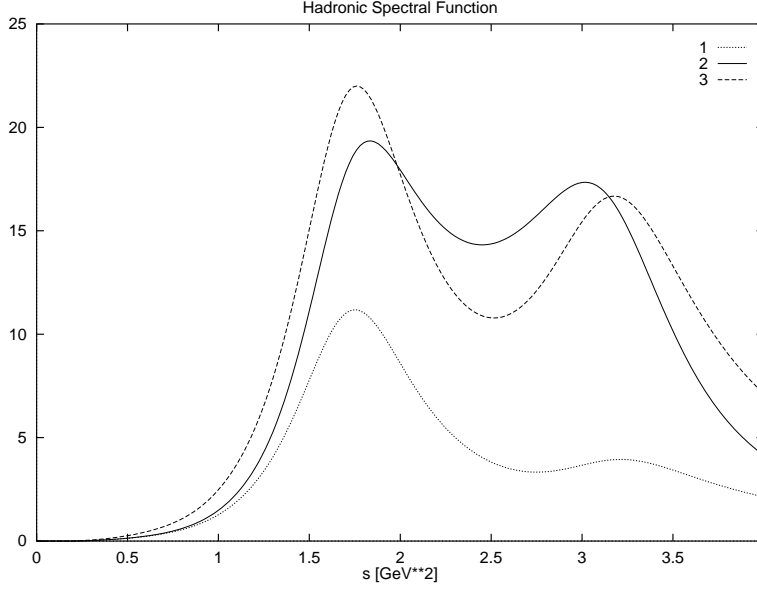


Figure 1: Shape of the hadronic continuum contribution to the spectral function in eq. (22) in units of 10^{-6} GeV^4 . The continuous curve is the one which gives the best duality with QCD. The dotted curve is the one which reproduces the best fit to the experimental curves which observe the $\pi'(1770)$ in hadronic interactions. The dashed curve is the one with the best duality when the normalization at the 3π -threshold is twice as large as in eq. (33).

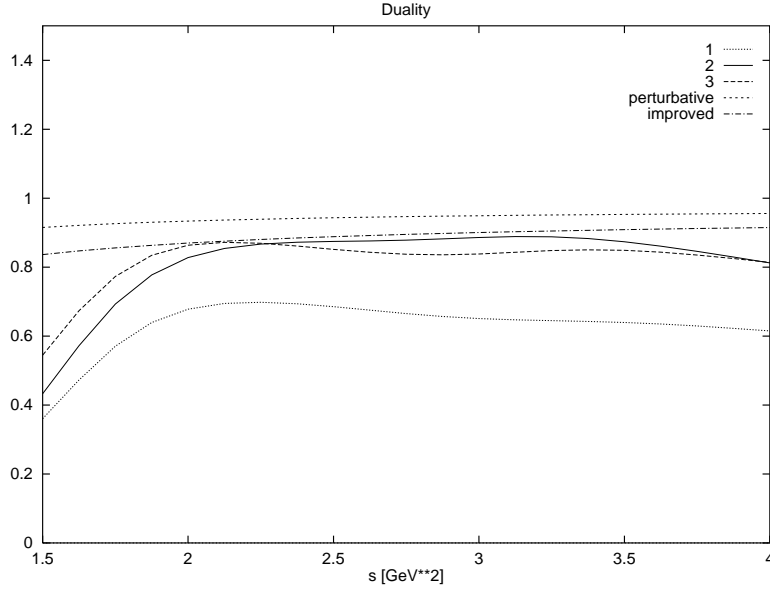


Figure 2: QCD-Hadronic duality test. The short-dashed curve is the one corresponding to the QCD ratio $\mathcal{R}_{QCD}(s)$ in eq. (19) with $\nu^2 = s$. The dash-dotted curve is the “improved” ratio. The other curves are the ones corresponding to the hadronic ratio $\mathcal{R}_{had}(s)$ in eq. (18) using the same input values as in Fig. 1.

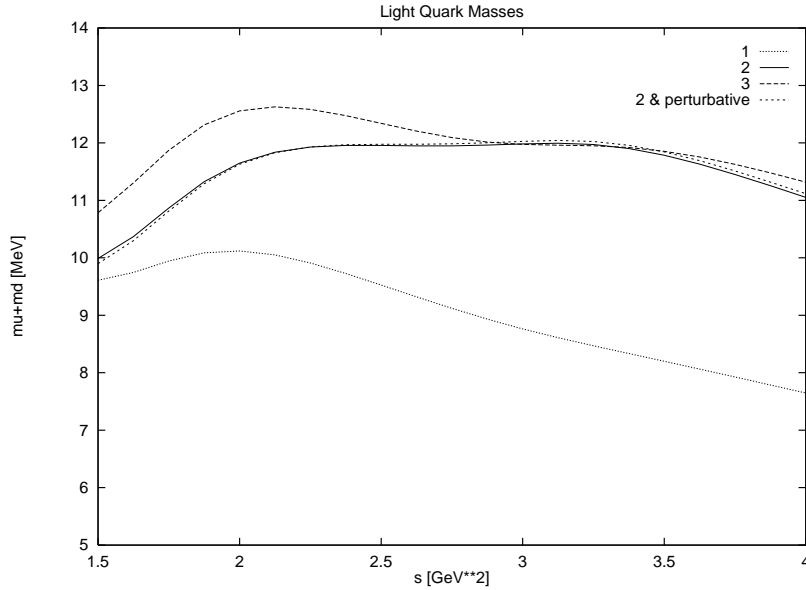


Figure 3: The running quark masses $m_u(1\text{GeV}^2) + m_d(1\text{GeV}^2)$ in the \overline{MS} -scheme. The three curves correspond to the results we obtain solving for $m_u + m_d$ in the FESR in eq. (12), using the “improved” expression for the QCD-function $R_1(s)$, and using the three hadronic parametrizations shown in Fig.1. The short-dashed curve shows the change if we use for the hadronic input the 2nd parametrization and the perturbative expression for the QCD counterpart.

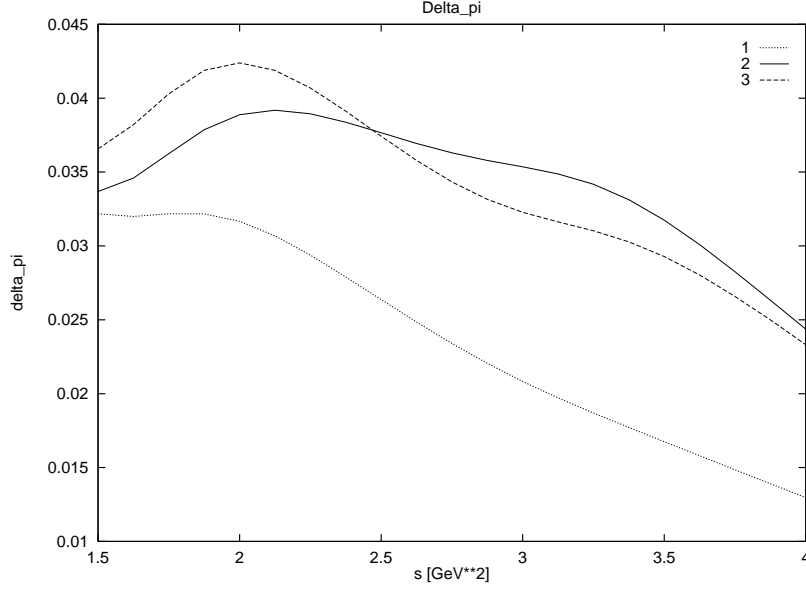


Figure 4: The parameter δ_π defined in eq.(9) which results from solving for δ_π in the first FESR in eq.(11). The three curves correspond to the hadronic parametrizations shown in Fig.1 and using the perturbative QCD expression for R_0 .

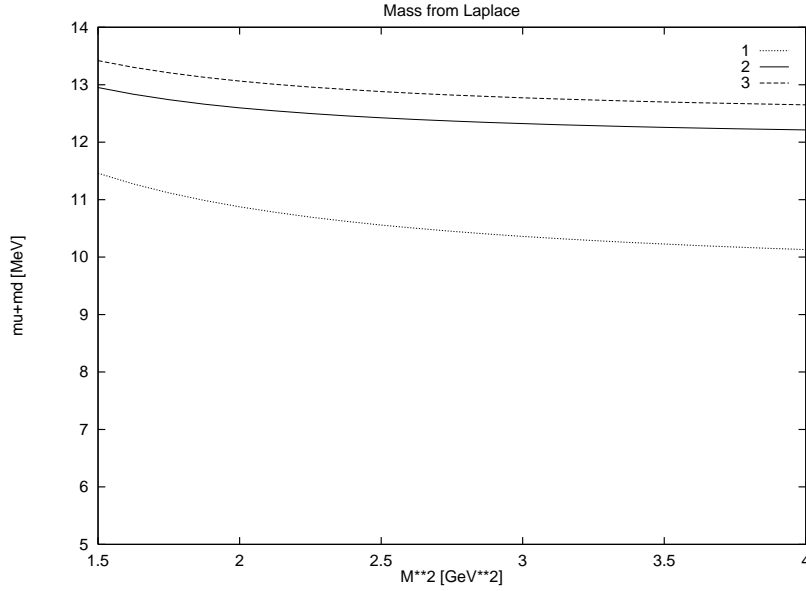


Figure 5: The running quark masses $m_u(1\text{GeV}^2) + m_d(1\text{GeV}^2)$ in the \overline{MS} -scheme, obtained with the Laplace transform sum rule in eq.(44). Curves 1,2 and 3 correspond to the hadronic parametrizations shown in Fig.1 and for $s = 2.5 \text{ GeV}^2$. The results for $s = 3.5 \text{ GeV}^2$ are very similar.